LIQUID DROP MODEL

This model was proposed by Neil Bohr during the year 1937. According to this model, a nucleus behaves like a charged liquid drop and the nucleons in the nucleus correspond to the molecules of a liquid in the drop. This model was based upon the following similarities between the nucleus and the liquid drop.

- The nucleus is considered as a drop of in compressible liquid of constant density but of varying mass.
- Both the nucleus and a drop of liquid are spherical in shape.
- The density of liquid is independent of size of the liquid drop. Similarly the density of nuclear matter is also independent of the size of the nucleus.
- The constant B.E. per nucleus is analogous to the latent heat of vaporization.
  [Note: The density of a nucleus is almost the same for nuclei of different elements but the density of a liquid is different for different liquids].
- The molecules in a liquid drop are held together by short-range inter-molecular forces, known as **cohesive forces** (S.T.). Similarly the nucleons in a nucleus are held together by short-range **nuclear forces**.
- The disintegration of nuclei by the emission of particles is an analogous to the evaporation of molecules from the surface of liquid.
- The formation of compound nucleus and absorption of bombarding particles correspond to the condensation of drops.

SEMI EMPIRICAL MASS FORMULA: The formula has been developed by Weizsacker's in 1935 by considering the different factors which affect nuclear binding. The mass M of a neutral atom whose nucleus contains Z protons and (A – Z) neutron is

$$M = ZM_p + (A - Z)M_n - B \quad ---------- (1)$$

Where B is the binding energy, which is made up of a number of terms each of which represents some general characteristic of nuclei i.e.,

$$\boldsymbol{B = B_0 + B_1 + B_2 + B_3 + B_4}$$

It is of great importance in nuclear stability problems. Now let us consider one by one.

I. **Volume energy:** The volume of a nucleus is proportional to its mass number A. This term may be regarded as volume **energy.** Larger the number of nucleons A, grater is the energy required to remove the individual protons and neutrons from the nucleus. Sometimes it is also called as

exchange energy. Therefore as the binding energy per nucleon is almost a constant, the volume energy is

$$\boldsymbol{B_0 = a_v A} \qquad ---(2)$$

Where a_v is proportionality constant.

II. **Surface energy:** The term B_0 represents the binding energy of nucleons which are totally within the nuclear volume. But we have to deduce a correction term for nucleons which are at the nuclear surface. In this case the nucleons at the surface of the nucleus are not completely surrounded by other nucleons. Hence the energy of the nucleons on the surface is less than that in the interior. Where as in the above expression all nucleons are being equally attracted on all sides. The radius of the nucleus is $R = R_0A^{1/3}$. Under the assumption of constant density

$$\therefore \boldsymbol{Surface\ area = 4\pi R^2}$$
$$\boldsymbol{= 4\pi R_0^2 A^{2/3}}$$

For light nuclei nearly all the nucleons are at the surface, whereas for heavy nuclei about half the nucleons are at the surface and half are in the interior of the nucleus. Thus we have to introduce a negative correction term B_1 representing the loss of binding energy by the nucleons at the surface.

$$\boldsymbol{B_1 = -a_s A^{2/3}} \qquad \dots\dots(2)$$ where a_s is a constant.

III. **Coulomb Energy:** The only long-range force in nuclei is the coulomb repulsion between protons. The coulomb repulsion is equal to the potential energy of Z-protons packed together in a spherically symmetric assembly of mean radius $R = R_0\ A^{1/3}$ and of uniform charge density $\frac{Ze}{\left(\frac{4}{3}\pi R_0^3\ A\right)}$. The coulomb energy is simply the work done against coulomb forces in assembling such a sphere. Thus the coulomb energy between the protons tends to lower the binding energy and its effect appears as a term with a negative sign. Therefore the loss of binding energy due to the disruptive coulomb energy is

$$\boldsymbol{B_2 = \frac{3}{5}\frac{e^2 Z^2}{R_0 A^{1/3}} = -a_c \frac{Z^2}{A^{1/3}}} \qquad \dots\dots(3)$$

Where a_c represents a constant and the subscript c designates coulomb energy.

IV. **Asymmetry Energy:** In general, in light nuclei, the number of neutrons is equal to number of protons (N=Z) where the coulomb effect is small thus these nuclei are highly stable. In heavy nuclei, the number of neutrons N > Z the number of protons. As (Z + N) increases, nuclear forces do not increase much, but repulsion increases with charge. Therefore proton increases less rapidly than neutrons, thus heavy nuclei have more neutrons than protons. Thus as the number of neutrons increases, the nucleus acquires an asymmetrical character due to which a force comes into play which reduces the volume energy. This decrease in energy is directly proportional to the square of the excess of neutrons over protons and inversely proportional to the total number of nucleons i.e. $B_3 \alpha \frac{(A-2Z)^2}{A}$

$$i.e. \quad \boldsymbol{B_3} = -\boldsymbol{a_a}\frac{(\boldsymbol{A}-\boldsymbol{2Z})^2}{\boldsymbol{A}} \qquad \dots\dots(\boldsymbol{4})$$

Where a_a is proportionality constant.

V. **Pairing Energy:** The nuclei with even number of protons and even number of neutrons are most stable. The nuclei with odd number of both neutrons and protons are the least stable, while nuclei for which either the proton or neutron number is even are inter mediate in stability. The pairing correction term is given as follows

$$B_4 = \begin{Bmatrix} +\delta \; for \; even - Z - even - N \\ -\delta \; for \; odd - Z - odd - N \\ \delta = 0 \; for \; odd \; A \end{Bmatrix}$$

Where δ is the correction term to the mass rather than the binding energy.

$$\boldsymbol{B_4} = \boldsymbol{\delta} = \pm\, \boldsymbol{a_p}\boldsymbol{A}^{-3/4} \dots\dots\dots\dots\dots\dots\dots.(\boldsymbol{5})$$

By considering the above results the Weizsacker's semi-empirical mass formula can be written as

$$\boldsymbol{M} = \boldsymbol{ZM_P} + (\boldsymbol{A}-\boldsymbol{Z})\boldsymbol{M_n} - \boldsymbol{a_v}\boldsymbol{A} + \boldsymbol{a_s}\boldsymbol{A}^{2/3} + \boldsymbol{a_c}\frac{\boldsymbol{Z}^2}{\boldsymbol{A}^{1/3}} + \boldsymbol{a_a}\frac{(\boldsymbol{A}-\boldsymbol{2Z})^2}{\boldsymbol{A}} \pm \boldsymbol{\delta}$$

This equation gives the expression for binding energy as well as atomic mass of a nucleus as a function of A and Z. The five constants of the formula can be determined by the combination of theoretical calculations and adjustment to fit experimental values of the masses (or binding energies).

The values of the constants have been determined with the results given below

(i) $a_v = 14Me, a_s = 13MeV, a_c = 0.595MeV, a_a = 19 < eV \; and \; a_p = 33.5MeV.$

But the liquid drop has failed to explain the measured spins and magnetic moments of nuclei.

Bohr and Wheeler's theory of Nuclear Fission:

On the basis of liquid drop model nuclear fission was explained by Bohr and Wheeler. According to this model the stable nucleus is considered as a spherical drop. The nucleons in the nucleus are held together by short range nuclear forces which are similar to the surface tension of the liquid. The shape of the drop depends upon the balance between surface tension force and coulomb repulsive forces. In the nucleus there are attractive forces between nucleons and repulsive forces between protons. When a uranium nucleus undergoes fission, it captures thermal or slow neutrons and a compound nucleus is formed. This compound nucleus is in a state of higher energy state and is set up in oscillations. These oscillations distort the shape of the spherical drop and become ellipsoidal in shape as shown in the figure. Now the restoring forces of the nucleus come into play from the short range inter-nuclear forces i.e. surface tension forces. These forces tries to maintain the shape of liquid spherical while the extra energy due to slow neutrons tends to distort the shape still further.

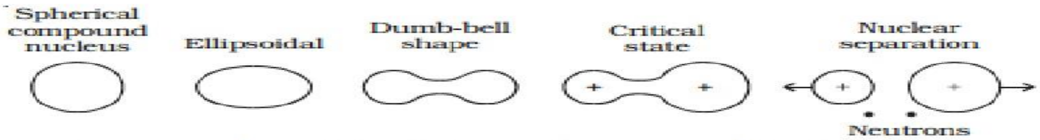


Fig Bohr – Wheeler's explanation of Nuclear fission

Suppose the oscillations are sufficiently large, the ellipsoid narrows and attains a dumb-bell shape as shown in Figure. If the oscillations are too large then the Colombian repulsive forces comes into play and pushes the two nuclear lopes of the dumbbell apart from each other. These two fragments are assumed to be spherical and positively charged. In this process two fragments are separated at a very high velocity accompanied by emission of some neutrons. These two fission fragments are not of the same size.

Merits and demerits of liquid drop model:

The liquid drop model has successfully explained a number of nuclear phenomena. Some of these important phenomena are.

(i) Constant density of nuclei, variation of B.E. per nucleon with mass number.

(ii) Explanation of intra-nuclear forces, Emission of particles during nuclear disintegration. It is similar to evaporation of molecules from the surface of liquid drop.

(iii) Fission of U^{235} by thermal neutrons and non-existence of nuclei heavier than U^{238} in nature.

But the liquid drop model has failed to explain the measured spins and magnetic moments of nuclei and also failed to explain magic numbers.

SHELL MODEL:

There are strong reasons to believe that as in the case of binding of the electrons in the atoms, the nucleons in nucleus are arranged in certain discrete shells.

W. M. Elasser, in 1933, was the first to point this out. Later, Maria Gopert Meyer (1948) and independently O Haxel, J.H.D. Jensen and H. E. Suess (1949) showed that the nuclei containing the following numbers of protons and neutrons exhibited very high stability:

Proton	**2**	**8**	**20**	**28**	**50**	**82**	**-**

Neutron	2	8	20	28	50	82	126

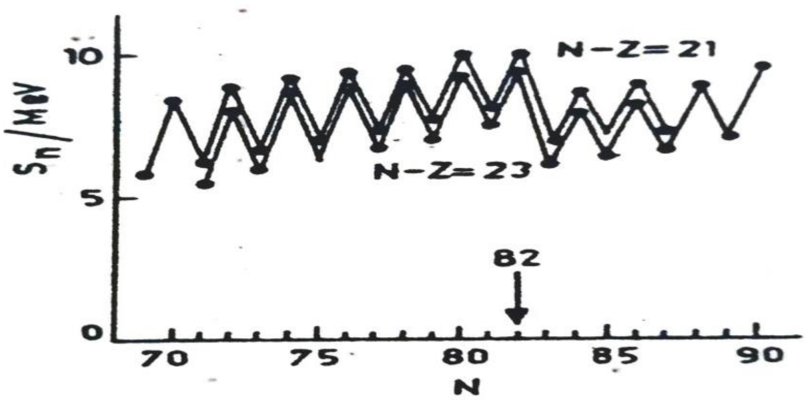
The above numbers are popularly known as **magic numbers** and are analogous to the atomic numbers of the inert gases. In addition to the above, there is a semi-magic number at N and Z = 40

Some nuclei contain magic numbers of protons and neutrons both. Examples ^{4}He (Z = 2, N = 2), ^{16}O (Z = 8, N = 8), ^{40}Ca (Z = 20, N = 20), ^{48}Ca (Z = 20, N = 28), ^{208}Pb (Z = 82, N = 126). They are doubly magic and show exceptionally high stability.

Following are the main evidences to show the existence of shell structure within the nuclei.

(a) Nuclei containing magic number of protons and neutrons show very high stability, compared to the nuclei containing one more nucleon of the same kind. Measurement shows that the separation energy S_n of a neutron from a nucleus containing a magic number of neutrons is large compared to that for a nucleus containing one more neutron. Similarly the separation energy Sp of a protons from a nucleus containing a magic number of protons is large compared to that for a nucleus containing one more proton. (Separation energy means the minimum energy needed for separating one neutron or proton from a nucleus).

The sudden discontinuity in the value of $S_n$at the magic neutron number 82 is shown in figure.



(b) The naturally occurring isotopes, whose nuclei contain magic numbers of neutrons or protons, have generally greater relative abundances (>60%).

For example, the isotopes ^{88}Sr (N = 50), ^{138}Ba (N = 82) and ^{140}Ce (N = 82) have relative abundances of 82.56%, 71.66% and 88.48% respectively.

(c) The number of stable isotopes of an element containing a magic number of protons is usually large compared to those for other elements.

For examples, calcium with Z=20 has 6 stable isotopes compared to 3 and 5 for argon (Z = 18) and titanium (Z = 22) respectively.

Again tin with Z = 50 has the largest number of stable isotopes. This number is 10 compared to 8 for cadmium (Z = 48) and tellurium (Z = 52).

(d) The number of naturally occurring isotones with magic number of neutrons is usually large compared to those in the immediate neighborhood.

As an example, the number of stable isotones at N = 82, is 7 compared to 3 and 2 at N = 80 and N = 84 respectively. Similar is the situation at N = 20, 28 and 50 which have respectively 5, 5 and 6 isotones. These numbers are greater than in the cases of the neighbouring isotones.

(e) The stable end products of all the three natural radioactive series are the three isotopes of lead (^{206}Pb, ^{207}Pb and ^{208}Pb) which all have the magic number Z = 82 of protons in their nuclei.

(f) Nuclei with magic numbers of neutrons or protons have their first excited states at higher energies than in the cases of the neighbouring nuclei.

(g) The neutron capture cross-sections of the nuclei with magic number of neutrons are usually low. Since the neutron shells are filled up in these nuclei, the probabilities of their capturing an additional neutron is small (see Fig.). Similarly nuclei with magic proton number have low proton capture cross-sections.

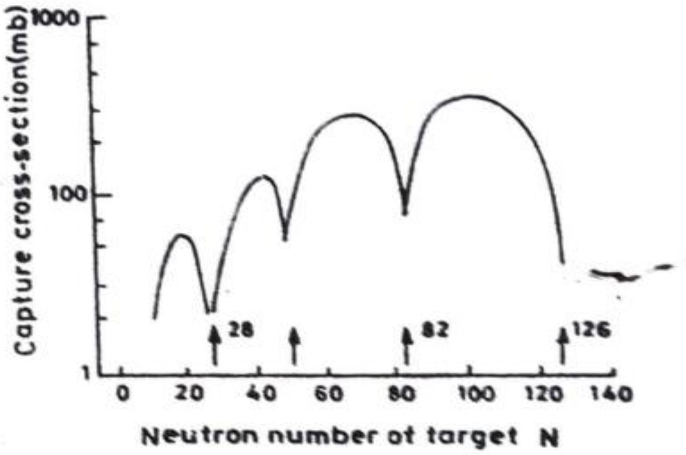


Fig. 9.7. Variation of neutron capture cross section with *N* showing discontinuities at the magic numbers.

(h) If the α-disintegration energies of the heavy nuclei are plotted as functions of the mass number A for a given Z, then usually a regular variation is observed till the magic neutron number N = 126 is reached when there is a sudden discontinuity (See Fig).

This confirms the magic character of the neutron number 126.

(i) Similar discontinuities are observed amongst the β-emitters at the magic neutron or proton numbers.

NUCLEAR FORCES: We know that nucleus consists of protons and neutrons. Proton is positively charges, hence there is an electrostatic force of repulsion between the protons, the gravitational force between the nucleons is too small to counter balance this repulsive force. Due to this repulsive force the nucleus should be highly unstable. But many nuclei are known to be stable. Naturally the question arises how these nucleons bound together with in the nucleus. The forces acting between the nucleons of the nucleus which keep them together are known as **nuclear forces**. The following are some of the characteristics of nuclear forces.

- Nuclear forces are strongest attractive force in nature; they are $\mathbf{10^{38}}$ times stronger than the gravitational force and **100** times the electrostatic force.

- They are very short range force and effective up to a distance of about 2-3 fermi. When the distance is less than 1 fermi there is a force of repulsion between the nucleons.
- **Nuclear forces are non central forces**: because the force between the two nucleons does not act along line joining their centers.
- **They are non-gravitational:** The gravitational force is a long range force. They are too weak to keep the nucleons intact in the nucleus.
- **Nuclear forces are charge independent**: nuclear forces acting between two neutrons or between two protons or between a proton and a neutron are of same nature. They do not depend on the charge of the particle. If fallows that the nuclear forces are non electric in nature.
- **Nuclear forces have saturation property**: Each nucleon can interact with only limited number of nucleons, very close to it but not far away from it. This effect is referred to as the saturation of nuclear forces.
- **Nuclear forces are spin dependent**: The force of attraction between two nucleons having parallel spin is stronger than the force between two nucleons having anti parallel spins.
- **Nuclear forces are exchange forces**: The nuclear force between two nucleons is due to the exchange of π meson between them.

Yukawa's Meson Theory of Nuclear Forces:

According to the meson theory of nuclear forces, all nucleons consist of identical cores surrounded by a "cloud" of one or more mesons. Mesons may be neutral or may have a positive or negative charge. The sole difference between neutrons and protons is supposed to lie in the composition of their respective meson clouds. Yukawa assumed that π meson is exchanged between the nucleons and that this exchange is responsible for the nuclear binding forces. The forces that act, between one neutron and another, and between one proton and another, are the result of the exchange of neutral mesons (π^0) between them. The force between a neutron and a proton is the result of the exchange of charged mesons ($\pi^+$and π) between them.

Thus a neutron emits a $\bar{\pi}$ meson and is converted into a proton:

$$\boldsymbol{n \rightarrow P + \bar{\pi}}$$

$$\boldsymbol{P + \bar{\pi} \rightarrow n}$$

In the reverse process, a proton emits a π^+ meson, becoming a neutron and the neutron, on receiving this π^+ meson, becomes a proton:

$$\boldsymbol{P \rightarrow n + \pi^+}$$

$$\boldsymbol{n + \pi^+ \rightarrow P}$$

Thus in the nucleus of an atom, attractive forces exist between (1) proton and proton (2) proton and neutron and (3) neutron and neutron. These forces of attraction are much larger than the electrostatic force of repulsion between the protons, thus giving a stability of the nucleus.

Just as a photon is a quantum of electromagnetic field, a meson is a quantum of nuclear field. Yukawa considered the equation for particle of mass m as,

$$\left(\nabla^2 - \frac{m^2c^2}{(h/2\pi)^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi = 0 \qquad \ldots.(1)$$

This is a relativistic equation valid only for spin less particles.

Separating the time dependent part, the equation for the radial part is

$$(\nabla^2 - \mu^2)\phi\,(r) = 0 \qquad \ldots.(2) \qquad where\ \mu = \frac{mc}{\frac{h}{2\pi}} = \frac{mc}{\hbar}$$

The solution of Eq. (2) is $\phi(r) = -g\frac{e^{-\mu r}}{r}$ $\ldots\ldots..(3)$

Here g is a constant, which plays the same role as the charge q in electromagnetic theory. In analogy with electromagnetism, the potential between two nucleons is then given by

$$V(r) = -g^2\frac{e^{-\mu r}}{r} \quad \ldots\ldots\ldots.(4)$$

Here g^2 is called the **'Coupling constant'**.

This argument made Yukawa predict the existence of pion as a quantum of nuclear force field.

The range of the pion field is $\frac{h/2\pi}{m_\pi C} \approx 1.4\ fm$.

The form of V (r) given by Eq. (4) is known as the **one-pion-exchange potential (OPEP).**

On the basis of the range of nuclear force and the uncertainty principle, it is possible to estimate the mass of the meson. According to uncertainty principle ΔE x Δt = h/2π where ΔE and Δt are the uncertainties in energy and time. The range of nuclear force is R≈ 1.4 x 10^{-15}m. Let us assume that the meson travels between nuclei at approximately the speed of light c. Let Δt be the time interval between the emission of meson from one nucleon and the absorption by the other nucleon.

$$Then \qquad \Delta t = R/c. \qquad \therefore\ \Delta E = \frac{(h/2\pi)}{\Delta t}$$

∴ The minimum meson mass is specified by $m \geq \frac{(h/2\pi)}{Rc}$.

In terms of the electronic mass m_e, the mass of the meson is

$$\frac{m}{m_e} = \frac{h/2\pi}{m_e\ Rc} = \frac{1.054\ x\ 10^{-34}}{(9.108\ x\ 10^{-31})(1.4\ x10^{-15})(3\ x\ 10^{8})} = 275$$

i.e., mass of the meson ≈ 275 x mass of electron or m=275m_e

NUCLEAR REACTIONS:

Introduction: When a collimated beam of intense and monochromatic light nuclear particle is bombarded upon a target material, the resulting interaction is known as **nuclear reaction.**

A nuclear reaction is a process in which one nuclide is converted into another nuclide or photon .In 1919, Rutherford conducted the first nuclear reaction in the laboratory by bombarding nitrogen with alpha particle emitted from a radioactive source**.**

$_7N^{14} + _2He^4 \rightarrow _8O^{17} + _1H^1$ It can be written as $_7N^{14}(\alpha,p)_8O^{17}$

According to Bohr, the general idea of nuclear reaction is

a + X → C→ Y+b Or X(a,b)Y

Here 'a' is called projectile, 'b' is called ejectile, 'X' is the target nucleic, 'C' is compound nuclei and 'Y' is the residual nuclei.

Following are some of the main types of nuclear reactions:

(i) **Elastic Scattering:** In this case the incident particle strikes the target nucleus and leaves without loss of energy, but its direction may change.

Example: Scattering of α-particles from a thin gold foil.

$_{79}Au^{197} + _2He^4 \rightarrow _{79}Au^{197} + _2He^4$

The target nucleus remains unaffected.

(ii) **Inelastic Scattering:** In this case, the incident particle loses a part of its energy in exciting the target nucleus to higher allowed energy level. The excited nucleus later decays to the ground state, radiating the excess energy in the form of a γ-ray photon. Example:

$_3Li^7 + _1H^1 \rightarrow _3Li^{7*} + _1H^1$

$_3Li^{7*} \rightarrow _3Li^7 + \gamma$.

The nuclear reactions can be classified into two groups

i) Compound nuclear reactions

ii) Direct nuclear reactions

i) **Compound nuclear reactions:** In such reaction, the projectile is absorbed by the target nucleus forming a Compound nucleus which is in excited state. It is de-excited by emitting a gamma ray photon.

a + X → C*→ Y+b

ii) **Direct nuclear reactions:** In such type of nuclear reactions the passage of initial state to final state takes place only in one step. Direct nuclear reactions can be divided into three groups

a) Radiative Capture(Or Knock out reactions)

b) Disintegration (or Stripping reactions)

c) Photodisintegration (Or pickup reactions)

(a) Radiative Capture: Here the incident particle is captured by the target nucleus and the new nucleus is formed. The new nucleus, in general, has a considerable excess of energy and decays with the emission of one or more γ-ray photons. Example:

$_6C^{12} + {_1H^1} \rightarrow {_7N^{13*}} \rightarrow {_1N^{13}} + \gamma$

(b) Disintegration: Here the incident particle is absorbed by the target nucleus and the ejected particle is a different one. The composition of the resultant nucleus is also different from the parent nucleus. An example is the disintegration of beryllium by α-particle producing neutrons.

$_4Be^9 + {_2He^4} \rightarrow {_6C^{12}} + {_0n^1}$

(c) Photodisintegration: When target materials are bombarded with radiations, the resulting compound nuclei are usually formed in excited states. These nuclei generally get rid of the excess excitation energy through neutron emission. For example,

$_1H^2 + \gamma \rightarrow {_1H^1} + {_0n^1}$

This requires a photon of energy 2.225 MeV.

Conservation Laws: The conservation laws may be stated and illustrated by reference to some specific nuclear reaction, say

$_5B^{10} + {_2He^4} \rightarrow ({_1N^{14}}) \rightarrow {_1H^1} + {_6C^{13}} + Q$

i. **Conservation of charge:** Total charge is conserved in every type of nuclear reaction in (α,p) C^{13} there are seven protons initially, also seven in the compound nucleus and in the products of the reaction.

ii. **Conservation of nucleons:** The total number of nucleons entering and leaving the reaction is constant. In B^{10}(α,p) C^{13}, we find 14 nucleons at each stage of the reaction.

iii. **Conservation of Mass-**Energy: In nuclear reactions neither kinetic energy nor rest mass is conserved by itself. But their total is always conserved.

iv. **Conservation of parity:** The net parity before the reaction must equal the net parity after the reaction.

v. Linear momentum, angular momentum, spins and isotopic spin are the other physical quantities which are also conserved in a nuclear reaction.

Quantities not Conserved: The most prominent physical characteristics which are not conserved in nuclear reactions are the magnetic dipole moments and the electric quadrupole moments of the reacting nuclei. These moments depend upon the internal distribution of mass, charge and current within the nuclei involved and are not subject to conservation laws

The Q-value Equation for a Nuclear Reaction:

Let a particle of mass M_1 moving with velocity v_1 collide with a target nucleus of mass M_0 at rest (Figure). After collision, the particle O of mass M_3 is emitted with velocity v_3 at an angle θ and the recoiling nucleus P of mass M_2 is emitted with a velocity v_2 at an angleϕ.

The conservation of linear momentum in the plane of paper yields the equations

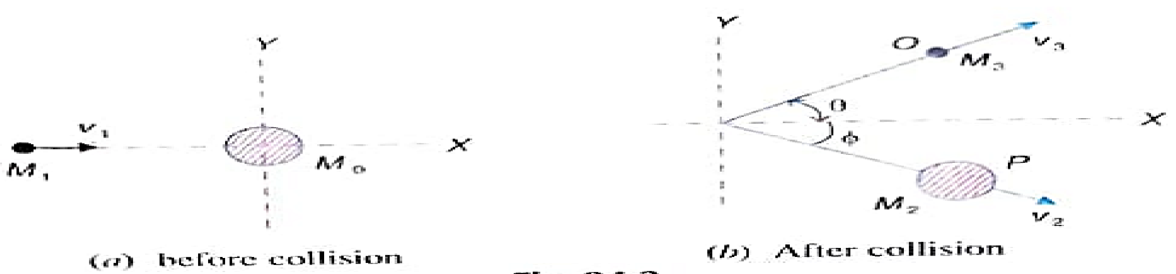


Fig. 34.2

Component of momentum along X-axis before and after collision

$$M_1 v_1 = M_3 v_3 \cos\theta + M_2 v_2 \cos\phi \qquad \ldots.(1)$$

Component of momentum along Y-axis before and after collision

$$0 = M_3 v_3 \cos\theta - M_2 v_2 \sin\phi \qquad \ldots\ldots.(2)$$

$$Equation\ (1) \Rightarrow \quad M_2 v_2 \cos\phi = M_1 v_1 - M_3 v_3 \cos\theta \qquad \ldots\ldots(3)$$

$$Equation\ (2) \Rightarrow M_2 v_2 \sin\phi = M_3 v_3 \sin\theta \qquad \ldots\ldots\ldots.(4)$$

Squaring and adding equations (3) and (4) we get

$$M_2^2 v_2^2 = M_1^2 v_1^2 + M_3^2 v_3^2 - 2M_1 M_3 v_1 v_3 \cos\theta \qquad \ldots\ldots.(5)$$

Using the kinetic energy relations,

$$E_{k_1} = \frac{1}{2} M_1 v_1^2, \qquad E_{k_2} = \frac{1}{2} M_2 v_2^2, \qquad and \quad E_{k_3} = \frac{1}{2} M_3 v_3^2$$

Substituting these values in Eq. (5), we have

$$2M_2 E_{k_2} = 2M_1 E_{k_1} + 2M_3 E_{k_3} - 2(M_1 M_3 E_{k_1} E_{k_3})^{1/2} 2\cos\theta$$

$$or \quad E_{k_2} = \frac{M_1 E_{k_1}}{M_2} + \frac{M_3 E_{k_3}}{M_2} - \frac{2}{M_2} (M_1 M_3 E_{k_1} E_{k_3})^{\frac{1}{2}} 2\cos\theta \qquad \ldots.(6)$$

$$Let \quad Q = \left(E_{k_2} + E_{k_3}\right) - E_{k_1} \qquad \ldots\ldots\ldots..(7)$$

Substituting the value of E_{k_2} from Eq. (6) in Eq. (7), we have

$$Q = \frac{M_1 E_{k_1}}{M_2} + \frac{M_3}{M_2} E_{k_3} - \frac{2}{M_2} (M_1 M_3 E_{k_1} E_{k_3})^{\frac{1}{2}} 2\cos\theta + E_{k_3} - E_{k_1}$$

$$Or \quad Q = E_{k_3}\left(1 + \frac{M_3}{M_2}\right) - E_{k_1}\left(1 - \frac{M_1}{M_2}\right) - \frac{2}{M_2} (M_1 M_3 E_{k_1} E_{k_3})^{\frac{1}{2}} 2\cos\theta \qquad \ldots..(8)$$

If θ=90°, cos 90° = 0. Hence, Eq. (8) reduces to

$$\boldsymbol{Q = E_{k_3}\left(1 + \frac{M_3}{M_2}\right) - E_{k_1}\left(1 - \frac{M_1}{M_2}\right) \quad \ldots\ldots\ldots\ldots.(9)}$$